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ASYMPTOTIC METHODS IN THE GRIFFITH PROBLEM*

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The problem of a crack in an elastic plane stretched at infinity is examined, taking sufficiently detailed account of the action of the interatomic cohesion forces between the crack edges. With respect to the functions characterizing the opening of the crack, the problem is reduced to a non-linear singular integrodifferential equation containing two dimension parameters. It is shown by using asymptotic methods of regular and matched asymptotic expansions that this equation has to solutions besides the trivial one. They correspond to cracks with respect to a small and a large opening. Fracture criteria are obtained by using the condition of smooth closing of the crack edges.

1. Let an infinite elastic body with a regular atomic lattice be loaded under plane strain conditions by uniform forces $\sigma_y = \text{const} \neq 0,\ \sigma_x = 0$. Then, it can be seen by taking Hooke's law into account that

$$\sigma_y = 2\theta\varepsilon_y, \qquad \theta = G(1-\nu)^{-1} \tag{1.1}$$

where ε_y is the relative elongation in the direction of the y axis, and G and ν are elastic constants. Under the mentioned loading of the body the spacing between rows of atoms in the crystal lattice will somewhat exceed the normal interatomic spacing b in the direction of the y axis, namely it will equal $b + \Delta b_y$, i.e., it can be assumed that $\varepsilon_y = \Delta b_y/b$. Furthermore, if the atomic series are separated, i.e., ε_y, is increased, then initially σ_y will increase proportional to the law (1.1), then for sufficiently large ε_y the linear relation will become non-linear, σ_y will reach a certain maximum value σ_p, the theoretical yield point, and then start to drop rapidly. Such a nature of the dependence of σ_y on ε_y is typical and corresponds to the Lennard-Jones, Morse, Massey, et al., interatomic interaction potentials and is shown in Fig.1. After reaching the value σ_p by the force σ_y, it must be considered that the continuity of the body is no longer preserved. We note that the described dependence of σ_y on ε_y will have the same qualitative nature even for amorphous bodies /1/.

Taking (1.1) into account, the dependence shown in Fig.1 will be written as

$$\sigma_y = 2\theta\varepsilon_y g(\varepsilon_y/d) \tag{1.2}$$

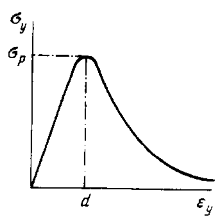

Fig.1

where the function $g(x)$ will decrease monotonically from the value $g(0) = 1$ to the value $g(\infty) = 0$ not more slowly than $x^{-\alpha}$ $(\alpha > 2)$, as x increases, and the quantity d will correspond to the strain ε_y for which σ_y reaches the maximum value σ_p. This means that the following relationships hold:

$$g(1) + g'(1) = 0, \qquad d = \sigma_p[2\theta g(1)]^{-1} = \delta/b \tag{1.3}$$

where δ is that excess over the normal interatomic distance for which the interaction between rows of atoms will start to drop for $\Delta b_y > \delta$. We find, on the basis of (1.2), that the work needed to form a unit of free surface equals /1, 2/

$$\gamma = \frac{b}{2}\int_d^\infty \sigma_y\, d\varepsilon_y = \frac{\delta\sigma_p I}{2g(1)} \quad \left(I = \int_1^\infty xg(x)\, dx\right) \tag{1.4}$$

We will now consider the classical problem of the tension in a plane with a crack of length $2a$ (plane strain) by uniform forces of intensity p applied at infinity (Fig.2). We introduce the opening of the crack $\Gamma(x) = -2v(x, 0)$ $(|x| < a)$, where $v(x, 0)$ is the displacement of points of the lower face of the crack. If "we view in a microscope" the crack edge $x = -a$,

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we will detect the separated rows of atoms (Fig.3). It must obviously be assumed that the crack starts where the spacing between the rows of atoms reaches a value of $b+\delta$ while the cohesive force between the atoms starts to drop. Therefore, the force σ_y increases monotonically during motion along the plane of the crack between $x=-\infty$ and $x=-a$, reaching the maximum value σ_p in the section $x=-a$, and then as x increases further starts to decrease monotonically (Fig.3). This means that at the crack tips $\Gamma(\pm a)=0$ and $\sigma_y(\pm a, 0)=\sigma_p$. The cohesive forces acting on the crack edges for $|x|\leqslant a$ (Fig.2) must be considered to be external forces relative to the deformable medium and introduced into the boundary conditions.

Thus, taking account of (1.2) and (1.3), the boundary conditions of the problem will have the form

$$y=0, \quad \tau_{xy}=0 \quad (|x|<\infty), \quad v=0 \quad (|x|>a) \tag{1.5}$$
$$\sigma_y=\frac{\sigma_p}{g(1)}\left(1+\frac{\Gamma}{\delta}\right)g\left(1+\frac{\Gamma}{\delta}\right) \quad (|x|\leqslant a)$$

where $\sigma_y=p$ at infinity. If the circumstance that the problem is physically non-linear in small zones near the crack tips by virtue of (1.2) is neglected, and the equations of linear elasticity theory are considered to hold everywhere in the elastic plane outside the crack, then by using the Fourier integral transform the problem of finding the solution of the Lamé equations for mixed boundary Conditions (1.5) can be reduced to the following non-linear singular integrodifferential equations (IDE):

$$\frac{\lambda}{\pi}\int_{-1}^{1}\frac{\Gamma'(\xi)}{\xi-x}\,d\xi=\frac{1}{g(1)}(1+\Gamma)g(1+\Gamma)-p \quad (|x|\leqslant 1,\ \Gamma(\pm 1)=0) \tag{1.6}$$

The following dimensionless quantities and notation are introduced here

$$\Gamma_*=\frac{\Gamma}{\delta}, \quad p_*=\frac{p}{\sigma_p}, \quad x_*=\frac{x}{a}, \quad \lambda=\frac{b}{4g(1)a}$$

We omit the asterisk in (1.6) and below.

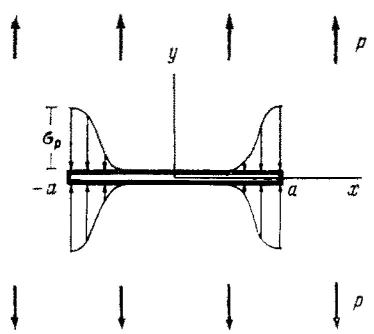

Fig.2

We will use the condition of smooth closure of the crack edges at its apices /3/

$$\Gamma'(\pm 1)=0 \tag{1.7}$$

to determine the critical value of the force p at which crack elongation will occur.

2. The non-linear Eq.(1.6) under Conditions (1.7) and for any value of λ has the trivial solution $\Gamma=0$, $p=1$ corresponding to the case of no crack opening and, consequently, fracture of the body on reaching the theoretical yield point. We will show that such a solution is obtained in practice only for sufficiently large values of the parameter λ.

Inverting the singular operator on the left-hand side of (1.6) under the conditions $\Gamma(\pm 1)=\Gamma'(\pm 1)=0$ we arrive at the IDE

$$\Gamma'(x)=-\frac{1}{\pi\lambda}\sqrt{1-x^2}\int_{-1}^{1}\frac{f(\Gamma)\,dt}{\sqrt{1-t^2}\,(t-x)}=A\Gamma \quad (|x|\leqslant 1, \Gamma(\pm 1)=0) \tag{2.1}$$

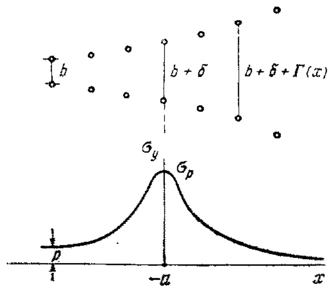

Fig.3

which is equivalent to (1.6) under the conditions

$$p = \frac{1}{\pi} \int_{-1}^{1} \frac{f(\Gamma)\,dt}{\sqrt{1-t^2}}, \quad f(\Gamma) = \frac{1}{g(1)}(1+\Gamma)\,g(1+\Gamma) \tag{2.2}$$

Integrating (2.1) taking the conditions $\Gamma(\pm 1) = 0$ into account, we find

$$\Gamma(x) = -\frac{1}{\pi\lambda} \int_{-1}^{x} \sqrt{1-\xi^2}\,d\xi \int_{-1}^{1} \frac{f(\Gamma)\,dt}{\sqrt{1-t^2}\,(t-\xi)} = B\Gamma \quad (|x| \leqslant 1) \tag{2.3}$$

Now assuming that $\Gamma(x) \in C_1(-1, 1)$, we make the estimate

$$\|A\Gamma\|_C \leqslant \frac{1}{\pi\lambda} \max_x \left| \int_{-1}^{1} \frac{\{f[\Gamma(t)] - f[\Gamma(x)]\}[\Gamma(t) - \Gamma(x)]}{[\Gamma(t) - \Gamma(x)]\sqrt{1-t^2}\,(t-x)}\,dt \right| < \frac{1}{\lambda} \max_\Gamma |f'(\Gamma)|\, \|\Gamma\|_{C_1} \tag{2.4}$$

Similarly we have

$$\|B\Gamma\|_C < {}^1/_2\pi\lambda^{-1} \max_\Gamma |f'(\Gamma)|\, \|\Gamma\|_{C_1} \tag{2.5}$$

Therefore, by virtue of (2.4) and (2.5)

$$\|B\Gamma\|_{C_1} < \lambda^{-1}(1 + {}^1/_2\pi) \max_\Gamma |f'(\Gamma)|\, \|\Gamma\|_{C_1} \tag{2.6}$$

and the operator B is a compression operator in $C_1(-1,1)$ for sufficiently large λ. In this case the solution of the integral Eq.(2.3) is unique in $C_1(-1,1)$ and can be obtained by successive approximations. However, one solution $\Gamma = 0$ is known and it is therefore unique, hence we have $p = 1$ from Conditions (2.2).

The further problem is to seek non-trivial solutions of (1.6) under the Conditions (1.7) for values of λ such that

$$\lambda < (1 + {}^1/_2\pi) \max_\Gamma |f'(\Gamma)| \tag{2.7}$$

It has been remarked /4/ that mathematical difficulties make a complete examination of (1.6) and (1.7) an extremely complex problem, but certain results can be obtained by asymptotic methods.

3. We will study the case of cracks of relatively small opening. To do this, we set

$$\Gamma = \lambda\Gamma^\circ, \qquad 1 - p = A\lambda^2 \tag{3.1}$$

in (1.6) and (1.7) and we seek the function $\Gamma^\circ(x)$ and the constant A in the form of the following regular expansions in powers of the small parameter λ:

$$\Gamma^\circ(x) = \Gamma_0(x) + \lambda\Gamma_1(x) + \lambda^2\Gamma_2(x) + O(\lambda^3) \tag{3.2}$$
$$A = A_0 + \lambda A_1 + \lambda^2 A_2 + O(\lambda^3)$$

Now substituting (3.1) into (1.6) and (1.7), using the smallness of λ and the first relationship in (1.3), we obtain, to terms of the order of λ^3,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma^{\circ\prime}(\xi)}{\xi-x}\,d\xi = A - B(\Gamma^\circ)^2 + \lambda C(\Gamma^\circ)^3 + \lambda^2 D(\Gamma^\circ)^4 \tag{3.3}$$

$$(|x|\leqslant 1,\ \Gamma^\circ(\pm 1) = \Gamma^{\circ\prime}(\pm 1) = 0)$$

$$B = 1 - \frac{g''(1)}{2g(1)},\quad C = \frac{g''(1) + {}^1\!/_3 g'''(1)}{2g(1)},\quad D = \frac{g'''(1) + {}^1\!/_4 g^{\mathrm{IV}}(1)}{6g(1)}$$

Taking account of the properties described above for the function $g(x)$ it can be shown that $B > 0$. Furthermore substituting the expansion (3.2) into (3.3) and equating terms on the right and left for identical powers of the parameter λ (up to λ^2 inclusive), we arrive at the following equations for the functions $\Gamma_i(x)$ $(i = 0, 1, 2)$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma_0'(\xi)}{\xi-x}\,d\xi = A_0 - B\Gamma_0^2(x) \tag{3.4}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma_1'(\xi)}{\xi-x}\,d\xi = A_1 - 2B\Gamma_0(x)\Gamma_1(x) + C\Gamma_0^3(x) \tag{3.5}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma_2'(\xi)}{\xi-x}\,d\xi = A_2 - 2B\Gamma_0(x)\Gamma_2(x) - B\Gamma_1^2(x) + 3C\Gamma_0^2(x)\Gamma_1(x) + D\Gamma_0^4(x) \tag{3.6}$$

$$(|x|\leqslant 1,\quad \Gamma_i(\pm 1) = \Gamma_i'(\pm 1) = 0)$$

The approximate solution of the non-linear IDE (3.4) can be found by the method of discrete vortices /5/, say, in combination with a quasilinearization process /6/. The integro-differential Eqs.(3.5) and (3.6) are Prandtl equations. Their approximate solutions can be constructed by one of the methods described in /5, 7/. The constants A_i $(i = 0, 1, 2)$ will be determined here from the conditions $\Gamma_i'(\pm 1) = 0$, equivalent to the following relationships (compare with (2.2)):

$$A_0 = \frac{B}{\pi}\int_{-1}^{1}\frac{\Gamma_0^2(t)}{\sqrt{1-t^2}}\,dt,\quad A_1 = \frac{1}{\pi}\int_{-1}^{1}\frac{2B\Gamma_0(t)\Gamma_1(t) - C\Gamma_0^3(t)}{\sqrt{1-t^2}}\,dt \tag{3.7}$$

$$A_2 = \frac{1}{\pi}\int_{-1}^{1}\frac{2B\Gamma_0(t)\Gamma_2(t) + B\Gamma_1^2(t) - 3C\Gamma_0^2(t)\Gamma_1(t) - D\Gamma_0^4(t)}{\sqrt{1-t^2}}\,dt$$

Therefore, the asymptotic solution of Eqs.(1.6) and (1.7) for small λ can actually be constructed in the form of (3.1) and (3.2). We find the critical force p from the second relationship of (3.1).

4. We will study the case of cracks of relatively larege opening. To do this we use the notation $\lambda/p = \mu$ and we set $\Gamma^1 = \mu\Gamma$. Then Eqs.(1.6), with the Condition (1.7), take the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma'(\xi)}{\xi-x}\,d\xi = \frac{\mu}{\lambda g(1)}\left(1 + \frac{\Gamma}{\mu}\right)g\left(1 + \frac{\Gamma}{\mu}\right) - 1 \tag{4.1}$$

$$(|x|\leqslant 1,\ \Gamma(\pm 1) = \Gamma'(\pm 1) = 0)$$

The supercript 1 is omitted from the Γ in (4.1) and henceforth. It is also possible to reduce (4.1) to the IDE

$$\Gamma'(x) = -\frac{\mu}{\pi\lambda}\sqrt{1-x^2}\int_{-1}^{1}\frac{f(\Gamma/\mu)\,dt}{\sqrt{1-t^2}\,(t-x)}\quad (|x|\leqslant 1,\ \Gamma(\pm 1) = 0) \tag{4.2}$$

equivalent to (4.1) under the conditions

$$1 = \frac{\mu}{\pi\lambda}\int\limits_{-1}^{1}\frac{f(\Gamma/\mu)\,dt}{\sqrt{1-t^2}}, \quad f\left(\frac{\Gamma}{\mu}\right) = \frac{1}{g(1)}\left(1+\frac{\Gamma}{\mu}\right)g\left(1+\frac{\Gamma}{\mu}\right) \tag{4.3}$$

Eqs.(4.2) and (4.3) are obviously analogous to (2.1) and (2.2).

We will apply the method of matched asymptotic expansions /8, 9/ to the investigation of (4.1), (4.2) and (4.3) for small values of the parameter μ. We have principally

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{\Gamma_0'(\xi)}{\xi - x}\,d\xi = -1 \quad (|x| \leqslant 1,\ \Gamma_0(\pm 1) = 0) \tag{4.4}$$

for determining the external (penetrating) solution

$$\Gamma_0(x) = \sqrt{1-x^2} \tag{4.5}$$

in the zone of variation of x outside of small neighbourhoods of the points $x = \pm 1$, where $\Gamma(x) \sim 1$ from (4.1) for $\mu \ll 1$ by virtue of the properties of the function $g(x)$.

We will now examine and ε-neighbourhood of the point $x = -1$ and we will extend it by introducing the new variable $r = (x+1)/\varepsilon$. The external solution (4.5) is of the order of $\varepsilon^{1/2}$ on approaching the ε-neighbourhood boundary, i.e., for $r \sim 1$. Consequently, we will seek the function $\Gamma(x)$ in the form

$$\Gamma(x) = \varepsilon^{1/2}q(r) + o(\varepsilon^{1/2})(r \sim 1) \tag{4.6}$$

in the ε-neighbourhood. Similarly, in the ε-neighbourhood of the point $x = 1$

$$\Gamma(x) = \varepsilon^{1/2}q(s) + o(\varepsilon^{1/2})\ (s = (1-x)/\varepsilon \sim 1) \tag{4.7}$$

Here $q(r)$ is an internal solution or boundary layer.

Substituting (4.6) and (4.7) into (4.2), we obtain in the neighbourhood of $x = -1$

$$\varepsilon^{-1/2}q'(r) = -\frac{\mu\varepsilon^{1/2}}{\pi\lambda}\sqrt{2r}\left(\int\limits_{-1}^{-1+\eta} + \int\limits_{-1+\eta}^{1-\eta} + \int\limits_{1-\eta}^{1}\right)\frac{f(\Gamma/\mu)\,dt}{\sqrt{1-t^2}\,(t-x)} \quad (\varepsilon \ll \eta \ll 1) \tag{4.8}$$

The integral between $-1+\eta$ and $1-\eta$ is small in (4.8) because the function $f(\Gamma/\mu) \lesseqgtr O(\mu^{\alpha-1})$ for $\Gamma \sim 1$ and $\mu \ll 1$. Moreover, it is seen that the third integral in (4.8) can also be neglected compared with the first (because $\varepsilon \ll 1$).

We set $\varepsilon^{1/2} = \mu$ to ensure matching of the external and internal solutions. Eq.(4.8) for $q(r)$ here takes the final form

$$q'(r) = -\frac{\varkappa}{\pi}\sqrt{2r}\int\limits_0^\infty \frac{f(q)\,d\zeta}{\sqrt{2\zeta}\,(\zeta - r)} \quad (0 \leqslant r < \infty,\ q(0) = 0) \tag{4.9}$$
$$\varkappa = \mu^2/\lambda \sim 1$$

The same equation is obtained if the ε-neighbourhood of the point $x = 1$ is examined.

By means of analogous reasoning the integral Condition (4.3) can be rewritten as follows:

$$1 = \frac{2\varkappa}{\pi}J, \quad J = \int\limits_0^\infty \frac{f(q)\,d\zeta}{\sqrt{2\zeta}} \tag{4.10}$$

We hence find the critical force

$$p = \sqrt{2J\lambda/\pi} \tag{4.11}$$

It is seen that it is determined principally by molecular interaction forces that appear in the ε-neighbourhoods of the crack tips.

5. Note that Condition (4.11) should obviously be exactly the same as the Griffith fracture condition (the energetic fracture condition) that has the form /2/ [2] $p = 2\sqrt{\theta\gamma/(\pi a)}$ in dimensional quantities. Taking account of (1.3), (1.4) and (1.7) in the dimensionless quantities mentioned earlier, this relationship can be rewritten as

$$p = 2\sqrt{I\lambda/(\pi g(1))} \tag{5.1}$$

Comparing (4.11) with (5.1) we see that the relationship

$$J = 2I/g(1) \tag{5.2}$$

should be satisfied, which imposes a constraint on the possible form of the function $g(x)$ in the original dependence (1.2). Therefore, the fracture micromechanism described by (1.2) turns out to be associated with the fracture macromechanism (5.1).

We will allow the function $g(x)$ in (1.2) to be selected such that the relationship (5.2) is satisfied approximately, and we will consider the problem of constructing an approximate solution of the non-linear IDE (4.9). For large values of the argument the function $q(r)$ should tend to the external solution, i.e., $q(r) = \sqrt{2r}\ (r \to \infty)$ according to (4.5) and (4.6). At the same time $q(0) = q'(0) = 0$ in the neighbourhood of zero. In accordance with this we will represent the approximate solution of (4.9) in the form

$$q(r) = (2r)^{3/2}(2r + D)^{-1} \tag{5.3}$$

where we select the constant D so that the relationship (5.2) is satisfied. Furthermore, taking (5.3) as the initial approximation for the solution of (4.9), its more exact solution can be found by the method of the discrete vortices /10/ and then a new expression can be found for the constant J according to (4.10). If it differs substantially from the value (5.2) taken earlier for J, then the form of the function $g(x)$ in (1.2) is inappropriate and must be corrected. Repeating this procedure several times, we finally find the form of the function $g(x)$ (generally not unique) needed to satisfy relationship (5.2) and an approximate solution of (4.9). Therefore, the asymptotic solution of (4.1) for small μ in the form (4.5), (4.6) and (4.9) can actually be constructed. We then find the critical force p from Condition (4.11) and (5.1).

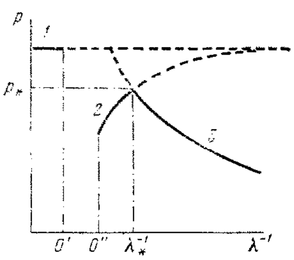

Fig.4

We now examine the plane p, λ^{-1}, where λ^{-1} is the dimensionless length of the crack (Fig.4). We schematically display the results obtained in this plane. The continuous part of curve *1* corresponds to the solution $p = 1$ (i.e., fracture before reaching the theoretical yield point) for cracks of quite small relative length that do not experience opening. Between the points O' and O'' of the axis λ^{-1} is a domain of cracks of small relative length $(\lambda \sim 1)$ for which information about the values of the critical force p can be obtained only by the direct numerical solution of (1.6) and (1.7). The solid part of curve *2* corresponds to the second relationship in (3.1) (it is taken into account that $A_0 > 0$, while the next terms in expansion (3.2) for $A(\lambda)$ are small for small λ). These are cracks of medium relative length and relatively small opening. The solid part of curve *3* corresponds to relation (5.1). These are cracks of long relative length and relatively large opening. From the nature of the dependence (3.1) and (5.1) it is seen that curves *2* and *3* should intersect at a certain point λ_*^{-1}. Therefore, cracks of medium relative length grow stably as the load acting on the body increases, but on reaching a certain critical length λ_*^{-1} for a load p_* less than the theoretical yield-point $p = 1$ they start to behave unstably according to the Griffith mechanism, like cracks of relative long length. The sections of the dependence $p = 1$ as well as (3.1) and (5.1) not realizable in actuality are shown by dashes in Fig.4.

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